

## ON LIMITING RELATIONS FOR CAPACITIES

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ABSTRACT. The paper is devoted to the study of limiting behaviour of Besov capacities  $\text{cap}(E; B_{p,q}^\alpha)$  ( $0 < \alpha < 1$ ) of sets in  $\mathbb{R}^n$  as  $\alpha \rightarrow 1$  or  $\alpha \rightarrow 0$ . Namely, let  $E \subset \mathbb{R}^n$  and

$$J_{p,q}(\alpha, E) = [\alpha(1 - \alpha)q]^{p/q} \text{cap}(E; B_{p,q}^\alpha).$$

It is proved that if  $1 \leq p < n$ ,  $1 \leq q < \infty$ , and the set  $E$  is open, then  $J_{p,q}(\alpha, E)$  tends to the Sobolev capacity  $\text{cap}(E; W_p^1)$  as  $\alpha \rightarrow 1$ . This statement fails to hold for compact sets. Further, it is proved that if the set  $E$  is compact and  $1 \leq p, q < \infty$ , then  $J_{p,q}(\alpha, E)$  tends to  $2n^p|E|$  as  $\alpha \rightarrow 0$  ( $|E|$  is the measure of  $E$ ). For open sets it is not true.

## 1. INTRODUCTION

The Sobolev space  $W_p^1(\mathbb{R}^n)$  ( $1 \leq p < \infty$ ) is defined as the class of all functions  $f \in L^p(\mathbb{R}^n)$  for which all first-order weak derivatives  $\partial f / \partial x_k = D_k f$  ( $k = 1, \dots, n$ ) exist and belong to  $L^p(\mathbb{R}^n)$ .

The classical embedding theorem with limiting exponent states that if  $1 \leq p < n$ , then for any  $f \in W_p^1(\mathbb{R}^n)$

$$\|f\|_{p^*} \leq c \sum_{k=1}^n \|D_k f\|_p, \quad \text{where } p^* = \frac{np}{n-p}. \quad (1.1)$$

This theorem was proved by Sobolev in 1938 for  $1 < p < n$  and by Gagliardo and Nirenberg in 1958 for  $p = 1$  (see [24, Chapter 5]).

Embeddings with limiting exponent are also true for some spaces defined in terms of moduli of continuity.

Let  $f \in L^p(\mathbb{R}^n)$  ( $1 \leq p < \infty$ ) and  $k \in \{1, \dots, n\}$ . The partial modulus of continuity of  $f$  in  $L^p$  with respect to  $x_k$  is defined by

$$\omega_k(f; \delta)_p = \sup_{0 \leq h \leq \delta} \left( \int_{\mathbb{R}^n} |f(x + he_k) - f(x)|^p dx \right)^{1/p}$$

( $e_k$  is the  $k$ th unit coordinate vector).

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Let  $0 < \alpha < 1$ ,  $1 \leq p < \infty$ ,  $1 \leq q \leq \infty$ , and  $k \in \{1, \dots, n\}$ . The Nikol'skiĭ-Besov space  $B_{p,q;k}^\alpha(\mathbb{R}^n)$  consists of all functions  $f \in L^p(\mathbb{R}^n)$  such that

$$\|f\|_{b_{p,q;k}^\alpha} \equiv \left( \int_0^\infty (t^{-\alpha} \omega_k(f; t)_p)^q \frac{dt}{t} \right)^{1/q} < \infty$$

if  $q < \infty$ , and

$$\|f\|_{b_{p,\infty;k}^\alpha} \equiv \sup_{t>0} t^{-\alpha} \omega_k(f; t)_p < \infty$$

if  $q = \infty$ . Further, set

$$B_{p,q}^\alpha(\mathbb{R}^n) = \bigcap_{k=1}^n B_{p,q;k}^\alpha(\mathbb{R}^n) \quad \text{and} \quad \|f\|_{b_{p,q}^\alpha} = \sum_{k=1}^n \|f\|_{b_{p,q;k}^\alpha}.$$

We write also  $B_{p,p}^\alpha(\mathbb{R}^n) = B_p^\alpha(\mathbb{R}^n)$ .

Observe that in these definitions and notations we follow Nikol'skiĭ's book [23]; they can be immediately extended to anisotropic Nikol'skiĭ-Besov spaces.

The spaces  $B_p^\alpha(\mathbb{R}^n)$  are often considered as Sobolev spaces of fractional smoothness. The embedding theorem with limiting exponent for these spaces asserts that if  $0 < \alpha < 1$  and  $1 \leq p < n/\alpha$ , then

$$B_p^\alpha(\mathbb{R}^n) \subset L^{p_\alpha}(\mathbb{R}^n), \quad \text{where} \quad p_\alpha = \frac{np}{n - \alpha p}. \quad (1.2)$$

This theorem was proved in the late sixties independently by several authors (for the references, see [4, § 18], [14, Section 10]).

In 2002 Bourgain, Brezis and Mironescu [6] discovered that embedding  $W_p^1 \subset L^{p^*}$  can be obtained as the limit of embedding (1.2) as  $\alpha \rightarrow 1$ . First, they proved in [5] that for any  $f \in W_p^1(\mathbb{R}^n)$  ( $1 \leq p < \infty$ )

$$\lim_{\alpha \rightarrow 1-} (1 - \alpha)^{1/p} \|f\|_{b_p^\alpha} \asymp \|\nabla f\|_p \quad (1.3)$$

(see also [7], [18, Section 14.3], [20, Section 10.2]). The main result in [6] is the following: if  $1/2 \leq \alpha < 1$  and  $1 \leq p < n/\alpha$ , then for any  $f \in B_p^\alpha(\mathbb{R}^n)$ ,

$$\|f\|_{L^{p_\alpha}}^p \leq c_n \frac{1 - \alpha}{(n - \alpha p)^{p-1}} \|f\|_{b_p^\alpha}^p \quad \left( p_\alpha = \frac{np}{n - \alpha p} \right), \quad (1.4)$$

where a constant  $c_n$  depends only on  $n$ . In view of (1.3), inequality (1.1) is a limiting case of (1.4) as  $\alpha \rightarrow 1-$ . The proof of (1.4) in [6] was quite complicated. Afterwards, Maz'ya and Shaposhnikova [21] gave a simpler proof of (1.4). Moreover, they studied the limiting behaviour

of the  $B_p^\alpha$ -norm and the sharp asymptotics of the embedding constant in (1.2) as  $\alpha \rightarrow 0$ . More precisely, they proved that

$$\|f\|_{L^{p_\alpha}}^p \leq c_{p,n} \frac{\alpha(1-\alpha)}{(n-\alpha p)^{p-1}} \|f\|_{b_p^\alpha}^p \quad \left(1 \leq p < \frac{n}{\alpha}, \quad p_\alpha = \frac{np}{n-\alpha p}\right). \quad (1.5)$$

Also, it was shown in [21] that if  $f \in B_p^{\alpha_0}(\mathbb{R}^n)$  for some  $\alpha_0 \in (0, 1)$ , then

$$\lim_{\alpha \rightarrow 0} \alpha \|f\|_{b_p^\alpha}^p \asymp \|f\|_p^p. \quad (1.6)$$

We note that in the works [6] and [21] a slightly different definition of the seminorm  $\|\cdot\|_{b_p^\alpha}$  was used; it is equivalent to the one given above.

Later on, it was observed in [17] that inequalities (1.4) and (1.5) can be directly derived from estimates of rearrangements obtained in [12].

Different extensions and some close aspects of these problems have been studied in [9], [10], [17], [19], [22], [25].

This paper was inspired by the results described above. Namely, it is devoted to the study of limiting behaviour of capacities in spaces  $B_{p,q}^\alpha$  as  $\alpha$  tends to 1 or  $\alpha$  tends to 0.

Let  $K \subset \mathbb{R}^n$  be a compact set. Denote by  $\mathfrak{N}(K)$  the set of all functions  $f \in C_0^\infty(\mathbb{R}^n)$  such that  $f(x) \geq 1$  for all  $x \in K$ . The capacity of the set  $K$  in the space  $W_p^1(\mathbb{R}^n)$  ( $1 \leq p < \infty$ ) is defined by

$$\text{cap}(K; W_p^1) = \inf \left\{ \left( \sum_{k=1}^n \|D_k f\|_p \right)^p : f \in \mathfrak{N}(K) \right\} \quad (1.7)$$

(see [20, 2.2.1]).

Similarly, let  $1 \leq p, q < \infty$  and  $0 < \alpha < 1$ . The capacity of a compact set  $K \subset \mathbb{R}^n$  in the space  $B_{p,q}^\alpha(\mathbb{R}^n)$  is defined by

$$\text{cap}(K; B_{p,q}^\alpha) = \inf \{ \|f\|_{b_{p,q}^\alpha}^p : f \in \mathfrak{N}(K) \} \quad (1.8)$$

(see [1], [2, Section 4], [20, Section 10.4]). Note that in this definition the  $p$ th power of the Besov norm is taken. This assures that the Hausdorff dimension of the set function  $\text{cap}(\cdot; B_{p,q}^\alpha)$  is equal to  $n - \alpha p$  when  $p < n/\alpha$  (see [1]).

Let  $X$  denote one of the spaces  $W_p^1(\mathbb{R}^n)$  or  $B_{p,q}^\alpha(\mathbb{R}^n)$ . Let  $G \subset \mathbb{R}^n$  be an open set. Then we define the capacity of  $G$  in  $X$  as

$$\text{cap}(G; X) = \sup \{ \text{cap}(K; X) : K \subset G, K \text{ is compact} \}.$$

The paper is organized as follows.

In Section 2 we give auxiliary statements which are used in the sequel.

In Section 3 we prove the main result of the paper. It states that if  $1 \leq p < n$  and  $1 \leq q < \infty$ , then for any open set  $G \subset \mathbb{R}^n$

$$\lim_{\alpha \rightarrow 1-} (1 - \alpha)^{p/q} \text{cap}(G; B_{p,q}^\alpha) = \left(\frac{1}{q}\right)^{p/q} \text{cap}(G; W_p^1). \quad (1.9)$$

We show that this statement may fail for a *compact* set. If  $n < p < \infty$ ,  $n \in \mathbb{N}$ , or  $n = p \geq 2$ , then equality (1.9) is trivially true because in these cases both the sides of (1.9) are equal to zero for any bounded open set  $G$ . Furthermore, (1.9) also trivially holds for  $p = n = 1$ ; in this case both the sides are equal to  $2q^{-1/q}$  for any non-empty open bounded set  $G \subset \mathbb{R}$ .

In Section 4 we consider the case  $\alpha \rightarrow 0$  and we prove that if  $1 \leq p, q < \infty$ , then for any compact set  $K \subset \mathbb{R}^n$

$$\lim_{\alpha \rightarrow 0+} \alpha^{p/q} \text{cap}(K; B_{p,q}^\alpha) = 2n^p \left(\frac{1}{q}\right)^{p/q} |K|$$

(as usual,  $|K|$  denotes the Lebesgue measure of  $K$ ). It is shown that generally this equality is not true for *open* sets.

## 2. AUXILIARY PROPOSITIONS

We begin with some properties of moduli of continuity.

We shall call *modulus of continuity* any non-decreasing, continuous and bounded function  $\omega(\delta)$  on  $[0, +\infty)$  which satisfies the conditions

$$\omega(\delta + \eta) \leq \omega(\delta) + \omega(\eta), \quad \omega(0) = 0. \quad (2.1)$$

It is well known that for any  $f \in L^p(\mathbb{R}^n)$  the functions  $\omega_j(f; \delta)_p$  are moduli of continuity.

For a modulus of continuity  $\omega$  the function  $\omega(\delta)/\delta$  may not be monotone. Therefore we shall use the following lemma.

**Lemma 2.1.** *Let  $\omega$  be a modulus of continuity. Set*

$$\overline{\omega}(t) = \frac{1}{t} \int_0^t \omega(u) du, \quad t > 0.$$

*Then*

$$\overline{\omega}(t) \leq \omega(t) \leq 2\overline{\omega}(t), \quad t > 0. \quad (2.2)$$

*Moreover,  $\overline{\omega}(t)$  increases and  $\overline{\omega}(t)/t$  decreases on  $(0, \infty)$ .*

*Proof.* Since

$$\overline{\omega}(t) = \int_0^1 \omega(tv) dv$$

and  $\omega$  is increasing, it is obvious that  $\bar{\omega}$  increases and the left-hand side inequality in (2.2) is true. We prove the right-hand side inequality in (2.2), that is,

$$\omega(t) \leq \frac{2}{t} \int_0^t \omega(u) du, \quad t > 0. \quad (2.3)$$

We have

$$\int_0^t \omega(u) du = \int_0^t \omega(t-u) du.$$

Thus, by (2.1),

$$2 \int_0^t \omega(u) du = \int_0^t [\omega(u) + \omega(t-u)] du \geq t\omega(t).$$

This implies (2.3). Using (2.3), we obtain

$$\left( \frac{\bar{\omega}(t)}{t} \right)' = -\frac{2}{t^3} \int_0^t \omega(u) du + \frac{\omega(t)}{t^2} \leq 0.$$

for almost all  $t > 0$ . Since  $\bar{\omega}(t)$  is locally absolutely continuous on  $(0, +\infty)$ , this implies that  $\bar{\omega}(t)/t$  decreases on  $(0, +\infty)$ .  $\square$

Now we consider some estimates of partial moduli of continuity.

First, it is obvious that for any  $f \in L^p(\mathbb{R}^n)$  ( $1 \leq p < \infty$ )

$$\omega_j(f; \delta)_p \leq 2 \|f\|_p \quad (j = 1, \dots, n). \quad (2.4)$$

It is easy to show that the constant 2 at the right-hand side is optimal (see Remark 4.3 below). However, for non-negative functions the constant can be improved. Namely, if  $f \in L^p(\mathbb{R}^n)$  and  $f(x) \geq 0$ , then

$$\omega_j(f; \delta)_p \leq 2^{1/p} \|f\|_p \quad (j = 1, \dots, n). \quad (2.5)$$

Indeed, let  $h > 0$ ,  $j \in \{1, \dots, n\}$ , and set  $E_{h,j} = \{x : f(x) \geq f(x + he_j)\}$ . Then

$$\begin{aligned} & \int_{\mathbb{R}^n} |f(x) - f(x + he_j)|^p dx \\ & \leq \int_{E_{h,j}} f(x)^p dx + \int_{\mathbb{R}^n \setminus E_{h,j}} f(x + he_j)^p dx \leq 2 \int_{\mathbb{R}^n} f(x)^p dx. \end{aligned}$$

This implies (2.5).

In what follows, for a set  $E \subset \mathbb{R}^n$  we denote by  $\chi_E$  its characteristic function. If  $E$  is a measurable set of finite measure, then by (2.5)

$$\omega_j(\chi_E; \delta)_p \leq (2|E|)^{1/p}. \quad (2.6)$$

If a function  $f \in L^p(\mathbb{R}^n)$  ( $1 \leq p < \infty$ ) has a weak derivative  $D_j f \in L^p(\mathbb{R}^n)$  for some  $1 \leq j \leq n$ , then

$$\omega_j(f; \delta)_p \leq \|D_j f\|_p \delta \quad (2.7)$$

(see [4, § 16]). Moreover, by the Hardy-Littlewood theorem [23, § 4.8], if  $1 < p < \infty$  and  $f \in L^p(\mathbb{R}^n)$ , then the relation  $\omega_j(f; \delta)_p = O(\delta)$  holds if and only if there exists the weak derivative  $D_j f \in L^p(\mathbb{R}^n)$ .

We shall also use the following well-known statement which we prove for completeness.

**Lemma 2.2.** *Let a function  $f \in L^p(\mathbb{R}^n)$  ( $1 \leq p < \infty$ ) have a weak derivative  $D_j f \in L^1_{loc} \mathbb{R}^n$  for some  $j \in \{1, \dots, n\}$ . Then*

$$\|D_j f\|_p = \lim_{\delta \rightarrow 0+} \frac{\omega_j(f; \delta)_p}{\delta} = \sup_{\delta > 0} \frac{\omega_j(f; \delta)_p}{\delta}. \quad (2.8)$$

*Proof.* The function  $f$  can be modified on a set of measure zero so that the modified function is locally absolutely continuous on almost all straight lines parallel to the  $x_j$ -axis, and its usual derivative with respect to  $x_j$  coincides almost everywhere on  $\mathbb{R}^n$  with  $D_j f$  (see [23, Chapter 4]). We assume that  $f$  itself has this property. Then

$$\frac{f(x + he_j) - f(x)}{h} \rightarrow D_j f(x) \quad \text{as } h \rightarrow 0$$

almost everywhere on  $\mathbb{R}^n$ . Thus, by Fatou's Lemma,

$$\begin{aligned} & \left( \int_{\mathbb{R}^n} |D_j f(x)|^p dx \right)^{1/p} \\ & \leq \liminf_{h \rightarrow 0+} \left( h^{-p} \int_{\mathbb{R}^n} |f(x + he_j) - f(x)|^p dx \right)^{1/p} \leq \liminf_{h \rightarrow 0+} \frac{\omega_j(f; h)_p}{h}. \end{aligned}$$

On the other hand, by (2.7)

$$\|D_j f\|_p \geq \sup_{h > 0} \frac{\omega_j(f; h)_p}{h} \geq \overline{\lim}_{h \rightarrow 0+} \frac{\omega_j(f; h)_p}{h}.$$

These inequalities yield (2.8).  $\square$

**Remark 2.3.** As we have observed above, for a modulus of continuity  $\omega$  the function  $\omega(\delta)/\delta$  may not be monotone. However, it is not difficult to show that for any modulus of continuity  $\omega$

$$\lim_{\delta \rightarrow 0+} \frac{\omega(\delta)}{\delta} = \sup_{\delta > 0} \frac{\omega(\delta)}{\delta}.$$

Now we derive some estimates involving Besov norms. First, we have the following lemma which we shall often use in the sequel.

**Lemma 2.4.** *Assume that a function  $f \in L^p(\mathbb{R}^n)$  ( $1 \leq p < \infty$ ) has a weak derivative  $D_j f \in L^p(\mathbb{R}^n)$  for some  $j \in \{1, \dots, n\}$ . Then  $f \in B_{p,q;j}^\alpha(\mathbb{R}^n)$  for any  $1 \leq q < \infty$  and any  $0 < \alpha < 1$ . Moreover,*

$$\|f\|_{b_{p,q;j}^\alpha} \leq q^{-1/q} [(1 - \alpha)^{-1/q} T^{1-\alpha} \|D_j f\|_p + 2\alpha^{-1/q} T^{-\alpha} \|f\|_p]$$

for any  $T > 0$ .

*Proof.* Applying estimates (2.4) and (2.7), we obtain for  $T > 0$

$$\begin{aligned} \|f\|_{b_{p,q;j}^\alpha} &\leq \left( \int_0^T t^{-\alpha q} \omega_j(f; t)_p^q \frac{dt}{t} \right)^{1/q} + \left( \int_T^\infty t^{-\alpha q} \omega_j(f; t)_p^q \frac{dt}{t} \right)^{1/q} \\ &\leq \|D_j f\|_p \left( \int_0^T t^{(1-\alpha)q} \frac{dt}{t} \right)^{1/q} + 2\|f\|_p \left( \int_1^\infty t^{-\alpha q} \frac{dt}{t} \right)^{1/q} \\ &= q^{-1/q} (1-\alpha)^{-1/q} T^{1-\alpha} \|D_j f\|_p + 2(\alpha q)^{-1/q} T^{-\alpha} \|f\|_p. \end{aligned}$$

□

It is well known that for fixed  $\alpha \in (0, 1)$  and  $p \in [1, \infty)$  the Besov spaces  $B_{p,q}^\alpha(\mathbb{R}^n)$  increase as the second index  $q$  increases. Moreover, the following estimate holds: if  $1 \leq p < \infty$ ,  $1 \leq q < \theta \leq \infty$ , and  $0 < \alpha < 1$ , then for any function  $f \in L^p(\mathbb{R}^n)$  and any  $j = 1, \dots, n$

$$\|f\|_{b_{p,\theta;j}^\alpha} \leq 8[\alpha(1-\alpha)]^{1/q-1/\theta} \|f\|_{b_{p,q;j}^\alpha} \quad (2.9)$$

(see [15, Lemma 2.2]). The constant coefficient at the right-hand side has optimal order as  $\alpha \rightarrow 1$  or  $\alpha \rightarrow 0$ . However, the value of this coefficient can be improved. First, for "small"  $\alpha$  we have the following result.

**Lemma 2.5.** *Let  $1 \leq p < \infty$ ,  $1 \leq q < \theta \leq \infty$ , and  $0 < \alpha < 1$ . Then for any function  $f \in L^p(\mathbb{R}^n)$  and any  $j = 1, \dots, n$*

$$\|f\|_{b_{p,\theta;j}^\alpha} \leq (\alpha q)^{1/q-1/\theta} \|f\|_{b_{p,q;j}^\alpha}. \quad (2.10)$$

*Proof.* Indeed, for any  $\delta > 0$  and any  $j \in \{1, \dots, n\}$ ,

$$\begin{aligned} \alpha \|f\|_{b_{p,q;j}^\alpha}^q &\geq \alpha \int_\delta^\infty t^{-\alpha q} \omega_j(f; t)_p^q \frac{dt}{t} \\ &\geq \omega_j(f; \delta)_p^q \alpha \int_\delta^\infty t^{-\alpha q} \frac{dt}{t} = \frac{1}{q} \delta^{-\alpha q} \omega_j(f; \delta)_p^q. \end{aligned}$$

Thus, we obtain (2.10) for  $\theta = \infty$ . From here, for any  $\theta \in (q, \infty)$ , we get

$$\begin{aligned} \|f\|_{b_{p,\theta;j}^\alpha}^\theta &= \int_0^\infty t^{-\alpha \theta} \omega_j(f; t)_p^\theta \frac{dt}{t} \\ &\leq \|f\|_{b_{p,\infty;j}^\alpha}^{\theta-q} \int_0^\infty t^{-\alpha q} \omega_j(f; t)_p^q \frac{dt}{t} \leq (\alpha q)^{(\theta-q)/q} \|f\|_{b_{p,q;j}^\alpha}^\theta. \end{aligned}$$

This yields (2.10). □

The following lemma plays an essential role in the case  $\alpha \rightarrow 1 - 0$ .

**Lemma 2.6.** *Let  $1 \leq p < \infty$ ,  $1 \leq q < \theta \leq \infty$ , and  $0 < \alpha < 1$ . Then for any function  $f \in L^p(\mathbb{R}^n)$  and any  $j = 1, \dots, n$*

$$\|f\|_{b_{p,\theta;j}^\alpha} \leq [(1-\alpha)q]^{1/q-1/\theta} \left( \frac{2}{1+\alpha} \right)^{1-q/\theta} \|f\|_{b_{p,q;j}^\alpha}. \quad (2.11)$$

*Proof.* Fix  $j \in \{1, \dots, n\}$  and set

$$\overline{\omega}(t) = \frac{1}{t} \int_0^t \omega_j(f; u)_p du, \quad t > 0.$$

By Hardy's inequality [3, p. 124],

$$\int_0^\infty t^{-\alpha q} \overline{\omega}(t)^q \frac{dt}{t} \leq \frac{1}{(1+\alpha)^q} \int_0^\infty t^{-\alpha q} \omega_j(f; t)_p^q \frac{dt}{t}.$$

Using this estimate, we have

$$\begin{aligned} \|f\|_{b_{p,q;j}^\alpha}^q &= \int_0^\infty t^{-\alpha q} \omega_j(f; t)_p^q \frac{dt}{t} \\ &\geq (1+\alpha)^q \int_0^\infty t^{-\alpha q} \overline{\omega}(t)^q \frac{dt}{t} \\ &\geq (1+\alpha)^q \int_0^\delta t^{(1-\alpha)q} \left( \frac{\overline{\omega}(t)}{t} \right)^q \frac{dt}{t} \end{aligned}$$

for any  $\delta > 0$ . By Lemma 2.1,  $\overline{\omega}(t)/t$  decreases on  $(0, +\infty)$ . Hence,

$$\begin{aligned} (1-\alpha) \|f\|_{b_{p,q;j}^\alpha}^q &\geq (1+\alpha)^q (1-\alpha) \left( \frac{\overline{\omega}(\delta)}{\delta} \right)^q \int_0^\delta t^{(1-\alpha)q} \frac{dt}{t} \\ &= \frac{(1+\alpha)^q}{q} \delta^{-\alpha q} \overline{\omega}(\delta)^q, \quad \delta > 0. \end{aligned}$$

By (2.2),  $\omega_j(f; \delta)_p \leq 2\overline{\omega}(\delta)$ , and thus we obtain

$$(1-\alpha) \|f\|_{b_{p,q;j}^\alpha}^q \geq \frac{1}{q} \left( \frac{1+\alpha}{2} \right)^q \delta^{-\alpha q} \omega_j(f; \delta)_p^q, \quad \delta > 0.$$

This implies inequality (2.11) for  $\theta = \infty$ . In the case  $\theta < \infty$  this inequality follows as in the proof of Lemma 2.5.  $\square$

Next, we consider some estimates of *distribution functions*.

For any measurable function  $f$  on  $\mathbb{R}^n$ , denote

$$\lambda_f(y) = |\{x \in \mathbb{R}^n : |f(x)| > y\}|, \quad y > 0.$$

Let  $S_0(\mathbb{R}^n)$  be the class of all measurable and almost everywhere finite functions  $f$  on  $\mathbb{R}^n$  such that  $\lambda_f(y) < \infty$  for each  $y > 0$ .



A *non-increasing rearrangement* of a function  $f \in S_0(\mathbb{R}^n)$  is a non-increasing function  $f^*$  on  $(0, +\infty)$  such that for any  $y > 0$

$$|\{t > 0 : f^*(t) > y\}| = \lambda_f(y).$$

We shall assume in addition that the rearrangement  $f^*$  is left continuous on  $(0, \infty)$ . Under this condition it is defined uniquely by

$$f^*(t) = \inf\{y > 0 : \lambda_f(y) < t\}, \quad 0 < t < \infty.$$

It follows that

$$f^*(\lambda_f(y)) \geq y \quad \text{for any } y \geq 0. \quad (2.12)$$

Set also

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(u) du.$$

For any  $f \in S_0(\mathbb{R}^n)$

$$f^{**}(t) = \int_t^\infty \frac{f^{**}(u) - f^*(u)}{u} du, \quad t > 0. \quad (2.13)$$

If  $f \in S_0(\mathbb{R}^n)$  is locally integrable and has all weak derivatives  $D_k f \in L^1_{\text{loc}}$  ( $k = 1, \dots, n$ ), then

$$f^{**}(t) - f^*(t) \leq n t^{1/n} \sum_{k=1}^n (D_k f)^{**}(t) \quad (t > 0) \quad (2.14)$$

(see [13, Lemma 5.1], [16, Lemma 3.1]).

**Lemma 2.7.** *Let  $f \in W_p^1(\mathbb{R}^n)$ ,  $1 \leq p < n$ , and let  $p^* = np/(n-p)$ . Then*

$$\lambda_f(y) \leq c_{p,n} \left( \sum_{k=1}^n \|D_k f\|_p \right)^{p^*} y^{-p^*}, \quad y > 0. \quad (2.15)$$

*Proof.* Of course, this weak-type inequality follows from the strong-type inequality (1.1). However, (2.15) is a direct consequence of the estimate (2.14). Indeed, by (2.13) and (2.14),

$$\begin{aligned} f^*(t) &\leq f^{**}(t) \leq n \sum_{k=1}^n \int_t^\infty u^{1/n-1} (D_k f)^{**}(u) du \\ &= nn' \sum_{k=1}^n \left[ t^{1/n-1} \int_0^t (D_k f)^*(u) du + \int_t^\infty u^{1/n-1} (D_k f)^*(u) du \right]. \end{aligned}$$

Applying Hölder inequality to both the integrals at the right-hand side, we have

$$f^*(t) \leq c t^{-1/p^*} \sum_{k=1}^n \|D_k f\|_p.$$

Setting  $t = \lambda_f(y)$  and taking into account (2.12), we get (2.15).  $\square$

Similarly, estimates of distribution functions in terms of moduli of continuity can be derived from the following inequality: for any  $f \in L^p(\mathbb{R}^n)$  ( $1 \leq p < \infty$ )

$$f^{**}(t) - f^*(t) \leq 2t^{-1/p} \sum_{k=1}^n \omega_k(f; t^{1/n})_p. \quad (2.16)$$

This inequality was first proved by Ul'yanov [26] in the one-dimensional case (see [14, p. 148] for an alternative proof). For all  $n \geq 1$  it was proved in [11]; a simpler proof was given in [12, Theorem 1].

**Lemma 2.8.** *Let  $0 < \alpha < 1$ ,  $1 \leq p < n/\alpha$ , and  $p_\alpha = np/(n - \alpha p)$ . Then for any function  $f \in L^p(\mathbb{R}^n)$*

$$\lambda_f(y) \leq (2p_\alpha)^{p_\alpha} \|f\|_{b_{p,\infty}^\alpha}^{p_\alpha} y^{-p_\alpha}, \quad y > 0. \quad (2.17)$$

*Proof.* We have

$$\sum_{k=1}^n \omega_k(f; t)_p \leq t^\alpha \|f\|_{b_{p,\infty}^\alpha} \quad \text{for any } t \geq 0.$$

Thus, by (2.13) and (2.16),

$$\begin{aligned} f^*(t) &\leq f^{**}(t) \leq 2 \sum_{k=1}^n \int_t^\infty u^{-1/p} \omega_k(f; u^{1/n})_p \frac{du}{u} \\ &\leq 2 \|f\|_{b_{p,\infty}^\alpha} \int_t^\infty u^{-1/p+\alpha/n} \frac{du}{u} = 2p_\alpha \|f\|_{b_{p,\infty}^\alpha} t^{-1/p_\alpha}. \end{aligned}$$

Setting  $t = \lambda_f(y)$  and applying (2.12), we obtain (2.17).  $\square$

We shall use the following notations. For any  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$  we denote by  $\hat{x}_k$  the  $(n-1)$ -dimensional vector obtained from the  $n$ -tuple  $x$  by removal of its  $k$ th coordinate. Let  $E \subset \mathbb{R}^n$ . For every  $k = 1, \dots, n$ , denote by  $\Pi_k(E)$  the orthogonal projection of  $E$  onto the coordinate hyperplane  $x_k = 0$ . If  $E$  is a set of the type  $F_\sigma$ , then all its projections  $\Pi_k(E)$  are sets of the type  $F_\sigma$  in  $\mathbb{R}^{n-1}$  and therefore they are measurable in  $\mathbb{R}^{n-1}$ . The  $(n-1)$ -dimensional measure of the projection  $\Pi_k(E)$  will be denoted by  $\text{mes}_{n-1} \Pi_k(E)$ . For the  $n$ -dimensional measure of the set  $E$  we keep the usual notation  $|E|$ . As above, by  $e_k$  we denote the  $k$ th unit coordinate vector.

**Lemma 2.9.** *Let  $\mu$ ,  $\lambda$ , and  $\eta$  be positive numbers and let  $n \in \mathbb{N}$ . Then for any set  $E \subset \mathbb{R}^n$  of the type  $F_\sigma$ , satisfying the conditions*

$$|E| \leq \mu \quad \text{and} \quad \text{mes}_{n-1} \Pi_k(E) \geq \lambda \quad (k = 1, \dots, n), \quad (2.18)$$

there exists  $0 < h \leq 2\mu^2 n / (\lambda\eta)$  such that

$$\sum_{k=1}^n |\{x \in E : x + he_k \in E\}| < \eta. \quad (2.19)$$

*Proof.* Let  $E \subset \mathbb{R}^n$  satisfy (2.18). Denote

$$\varphi_{E,k}(h) = |\{x \in E : x + he_k \in E\}| = \int_E \chi_E(x + he_k) dx \quad (h > 0).$$

For any  $H > 0$  and any  $k = 1, \dots, n$ , we have

$$\begin{aligned} \int_0^H \varphi_{E,k}(h) dh &= \int_E dx \int_0^H \chi_E(x + he_k) dh \\ &\leq |E| \int_{\mathbb{R}} \chi_E(y) dy_k. \end{aligned}$$

Integrating over projection  $\Pi_k(E)$ , we obtain

$$\begin{aligned} \text{mes}_{n-1} \Pi_k(E) \int_0^H \varphi_{E,k}(h) dh \\ \leq |E| \int_{\Pi_k E} d\hat{y}_k \int_{\mathbb{R}} \chi_E(y) dy_k = |E|^2. \end{aligned}$$

By (2.18), this implies that

$$\int_0^H \varphi_{E,k}(h) dh \leq \frac{\mu^2}{\lambda} \quad (k = 1, \dots, n).$$

Denoting

$$\varphi_E(h) = \sum_{k=1}^n \varphi_{E,k}(h),$$

we have

$$\int_0^H \varphi_E(h) dh \leq \frac{\mu^2 n}{\lambda}.$$

Thus,

$$\inf_{h \in [0, H]} \varphi_E(h) \leq \frac{\mu^2 n}{\lambda H}$$

Setting  $H = (2\mu^2 n / (\lambda\eta))$ , we obtain that there exists  $h \in (0, H]$  (depending on  $\mu$ ,  $\lambda$ ,  $\eta$ , and  $E$ ) such that  $\varphi_E(h) < \eta$ .  $\square$

Throughout this paper  $\mathcal{B}_r$  denotes the open ball with radius  $r > 0$  centered at the origin. In the sequel we shall use *the standard mollifier* (see, e.g., [18, p. 553])

$$\varphi(x) = \begin{cases} c \exp(1/(|x|^2 - 1)) & \text{if } x \in \mathcal{B}_1 \\ 0 & \text{if } x \notin \mathcal{B}_1, \end{cases} \quad (2.20)$$

where  $c > 0$  is such that

$$\int_{\mathbb{R}^n} \varphi(x) dx = 1.$$

Set for  $\tau > 0$

$$\varphi_\tau(x) = \frac{1}{\tau^n} \varphi\left(\frac{x}{\tau}\right). \quad (2.21)$$

Then  $\varphi_\tau(x) = 0$  if  $|x| > \tau$ , and

$$\int_{\mathbb{R}^n} \varphi_\tau(x) dx = 1. \quad (2.22)$$

We shall also use the following *cutoff function*

$$\eta(x) = (\varphi * g)(x), \quad (2.23)$$

where  $g$  is the characteristic function of the open ball  $\mathcal{B}_2$ . We have that  $\eta \in C_0^\infty$ ,  $\eta(x) = 1$  if  $|x| \leq 1$  and  $\eta(x) = 0$  if  $|x| \geq 3$ .

Let  $f \in C^\infty(\mathbb{R}^n) \cap W_p^1(\mathbb{R}^n)$ . For any  $\gamma > 0$  the function  $f_\gamma(x) = f(x)\eta(\gamma x)$  belongs to  $C_0^\infty(\mathbb{R}^n)$ . Moreover, it is easy to see that for any  $\varepsilon > 0$  there exists  $\gamma_0 > 0$  such that for all  $0 < \gamma \leq \gamma_0$

$$\|D_k f_\gamma\|_p < \|D_k f\|_p + \varepsilon \quad (k = 1, \dots, n) \quad (2.24)$$

(see, e.g., [24, p. 124]).

In the sequel we use also the following remark concerning capacities. Let  $K \subset \mathbb{R}^n$  be a compact set. Denote by  $\mathfrak{P}(K)$  the set of all functions  $f \in C_0^\infty(\mathbb{R}^n)$  such that  $0 \leq f(x) \leq 1$  for all  $x \in \mathbb{R}^n$  and  $f(x) = 1$  in some neighborhood of  $K$ . It is well known that the set  $\mathfrak{N}(K)$  in definitions (1.7) and (1.8) may be replaced by  $\mathfrak{P}(K)$ . Namely,

$$\text{cap}(K; W_p^1) = \inf \left\{ \left( \sum_{k=1}^n \|D_k f\|_p \right)^p : f \in \mathfrak{P}(K) \right\}$$

and

$$\text{cap}(K; B_{p,q}^\alpha) = \inf \{ \|f\|_{b_{p,q}^\alpha}^p : f \in \mathfrak{P}(K) \}$$

(see [20, 2.2.1]).

### 3. THE LIMIT AS $\alpha \rightarrow 1$

In this section we prove the main result of the paper. As we have already mentioned in the Introduction, this result was inspired by the limiting relation (1.3) proved in [5]. We observe that the following slight modification of (1.3) holds: if a function  $f \in L^p(\mathbb{R}^n)$  has a weak

derivative  $D_j f \in L^p(\mathbb{R}^n)$ , then  $f \in B_{p,q;j}^\alpha(\mathbb{R}^n)$  for any  $1 \leq q < \infty$  and any  $0 < \alpha < 1$ , and

$$\lim_{\alpha \rightarrow 1-0} (1-\alpha)^{1/q} \|f\|_{b_{p,q;j}^\alpha} = \left(\frac{1}{q}\right)^{1/q} \|D_j f\|_p.$$

This statement follows by standard arguments from Lemma 2.2 and inequality (2.4) (see also [18, Section 14.3]).

**Theorem 3.1.** *Let  $n \geq 2$ ,  $1 \leq p < n$ , and  $1 \leq q < \infty$ . Then for any open set  $G \subset \mathbb{R}^n$*

$$\lim_{\alpha \rightarrow 1-0} (1-\alpha)^{p/q} \text{cap}(G; B_{p,q}^\alpha) = \left(\frac{1}{q}\right)^{p/q} \text{cap}(G; W_p^1). \quad (3.1)$$

*Proof.* Denote

$$\Lambda(\alpha) = (1-\alpha)^{1/q} [\text{cap}(G; B_{p,q}^\alpha)]^{1/p}, \quad 0 < \alpha < 1. \quad (3.2)$$

First we shall show that

$$\overline{\lim}_{\alpha \rightarrow 1-0} \Lambda(\alpha) \leq q^{-1/q} [\text{cap}(G; W_p^1)]^{1/p}. \quad (3.3)$$

We assume that  $\text{cap}(G; W_p^1) < \infty$ . Let  $K \subset G$  be a compact set and let  $0 < \varepsilon < 1$ . There exists a function  $f \in C_0^\infty(\mathbb{R}^n)$  such that

$$\sum_{k=1}^n \|D_k f\|_p < (\text{cap}(K; W_p^1) + \varepsilon)^{1/p}, \quad (3.4)$$

$0 \leq f(x) \leq 1$  for all  $x \in \mathbb{R}^n$ , and  $f(x) = 1$  in some neighborhood of  $K$ . Set  $E_\varepsilon = \{x : f(x) > \varepsilon\}$ . By Lemma 2.7,

$$|E_\varepsilon| \leq c_{p,n} \left( \sum_{k=1}^n \|D_k f\|_p \right)^{p^*} \varepsilon^{-p^*}, \quad p^* = \frac{np}{n-p}.$$

Using (3.4) and taking into account that  $K \subset G$ , we obtain that

$$|E_\varepsilon| \leq A \varepsilon^{-p^*}, \quad (3.5)$$

where  $A \equiv A(n, p, G) = c_{p,n} [\text{cap}(G; W_p^1) + 1]^{p^*/p}$ . We emphasize that  $A$  doesn't depend on  $K$ .

There exists an open set  $H$  such that  $K \subset H$  and  $f(x) = 1$  on  $H$ . Let  $\rho$  be the distance from  $K$  to the boundary of  $H$  and let  $0 < \tau < \rho/2$ . Set

$$f_\varepsilon(x) = \frac{1}{1-\varepsilon} \max(f(x) - \varepsilon, 0) \quad \text{and} \quad f_{\varepsilon,\tau}(x) = (f_\varepsilon * \varphi_\tau)(x),$$

where  $\varphi_\tau$  is defined by (2.21). Then  $f_\varepsilon \in W_p^1(\mathbb{R}^n)$  and

$$\|D_k f_\varepsilon\|_p \leq \frac{1}{1-\varepsilon} \|D_k f\|_p \quad (k = 1, \dots, n).$$

Furthermore,  $D_k f_{\varepsilon, \tau} = (D_k f_\varepsilon) * \varphi_\tau$ . Thus, by (2.22) and Young inequality,

$$\|D_k f_{\varepsilon, \tau}\|_p \leq \|D_k f_\varepsilon\|_p \leq \frac{1}{1-\varepsilon} \|D_k f\|_p \quad (k = 1, \dots, n). \quad (3.6)$$

It is clear that  $f_\varepsilon(x) = 0$  if  $x \notin E_\varepsilon$  and  $0 \leq f_\varepsilon(x) \leq 1$  for all  $x \in \mathbb{R}^n$ . First, by (2.22) and (3.5), this imply that

$$\|f_{\varepsilon, \tau}\|_p \leq \|f_\varepsilon\|_p \leq |E_\varepsilon|^{1/p} \leq (A\varepsilon^{-p^*})^{1/p}, \quad A = A(n, p, G). \quad (3.7)$$

We have also that  $0 \leq f_{\varepsilon, \tau}(x) \leq 1$  for all  $x \in \mathbb{R}^n$ . Furthermore,  $f_\varepsilon(x) = 1$  on  $H$ . This yields that  $f_{\varepsilon, \tau}(x) = 1$  for all  $x$  such that  $\text{dist}(x, K) < \tau$ . Indeed, if  $\text{dist}(x, K) < \tau$  and  $|y| \leq \tau$ , then  $x - y \in H$  and  $f_\varepsilon(x - y) = 1$ . Thus,

$$f_{\varepsilon, \tau}(x) = \int_{B_\tau} \varphi_\tau(y) f_\varepsilon(x - y) dy = 1.$$

Observe also that  $f_{\varepsilon, \tau} \in C_0^\infty(\mathbb{R}^n)$ . Taking into account these properties of  $f_{\varepsilon, \tau}$ , we have that

$$\text{cap}(K; B_{p, q}^\alpha) \leq \|f_{\varepsilon, \tau}\|_{b_{p, q}^\alpha}^p. \quad (3.8)$$

Applying Lemma 2.4 with  $T = 1$ , we obtain

$$(1 - \alpha)^{1/q} \|f_{\varepsilon, \tau}\|_{b_{p, q}^\alpha} \leq \left(\frac{1}{q}\right)^{1/q} \left[ \sum_{k=1}^n \|D_k f_{\varepsilon, \tau}\|_p + 2 \left(\frac{1 - \alpha}{\alpha}\right)^{1/q} \|f_{\varepsilon, \tau}\|_p \right].$$

Using (3.6) and (3.4) and taking into account that  $K \subset G$ , we have

$$\sum_{k=1}^n \|D_k f_{\varepsilon, \tau}\|_p \leq \frac{1}{1 - \varepsilon} [\text{cap}(G; W_p^1) + \varepsilon]^{1/p}.$$

The last two inequalities, together with (3.7) and (3.8), yield that

$$\begin{aligned} & (1 - \alpha)^{1/q} [\text{cap}(K; B_{p, q}^\alpha)]^{1/p} \\ & \leq \frac{1}{q^{1/q}(1 - \varepsilon)} [\text{cap}(G; W_p^1) + \varepsilon]^{1/p} + A' \left(\frac{1 - \alpha}{\alpha q}\right)^{1/q} \varepsilon^{-p^*/p}, \end{aligned}$$

where  $A' = 2A(n, p, G)^{1/p}$  doesn't depend on  $K$ . Taking supremum over all compact sets  $K \subset G$  and using notation (3.2), we get

$$\Lambda(\alpha) \leq \frac{1}{q^{1/q}(1 - \varepsilon)} [\text{cap}(G; W_p^1) + \varepsilon]^{1/p} + A' \left(\frac{1 - \alpha}{\alpha q}\right)^{1/q} \varepsilon^{-p^*/p}.$$

It follows that

$$\overline{\lim}_{\alpha \rightarrow 1-0} \Lambda(\alpha) \leq \frac{1}{q^{1/q}(1 - \varepsilon)} [\text{cap}(G; W_p^1) + \varepsilon]^{1/p}.$$

Since  $\varepsilon \in (0, 1)$  is arbitrary, this implies (3.3).

Now we shall prove that

$$\lim_{\alpha \rightarrow 1-0} \Lambda(\alpha) \geq q^{-1/q} [\text{cap}(G; W_p^1)]^{1/p}. \quad (3.9)$$

Let  $K \subset G$  be a compact set. Choose  $\tau > 0$  such that

$$K_\tau = \{x \in \mathbb{R}^n : \text{dist}(x, K) \leq 2\tau\} \subset G. \quad (3.10)$$

Then  $K_\tau$  is compact.

We assume that  $\lim_{\alpha \rightarrow 1-0} \Lambda(\alpha) < \infty$ . There exists an increasing sequence  $\{\alpha_\nu\}$  of numbers  $\alpha_\nu \in (0, 1)$  such that  $\alpha_\nu \rightarrow 1$  and

$$\lim_{\nu \rightarrow \infty} \Lambda(\alpha_\nu) = \lim_{\alpha \rightarrow 1-0} \Lambda(\alpha). \quad (3.11)$$

We assume also that

$$\Lambda(\alpha_\nu) \leq \lim_{\alpha \rightarrow 1-0} \Lambda(\alpha) + 1 \quad (\nu \in \mathbb{N}). \quad (3.12)$$

For any  $\nu \in \mathbb{N}$  there exists a function  $f_\nu \in C_0^\infty(\mathbb{R}^n)$  such that  $0 \leq f_\nu(x) \leq 1$  for all  $x \in \mathbb{R}^n$ ,  $f_\nu(x) = 1$  for all  $x \in K_\tau$ , and

$$\|f_\nu\|_{b_{p,q}^{\alpha_\nu}} \leq \text{cap}(K_\tau; B_{p,q}^\alpha)^{1/p} + \frac{1}{\nu}.$$

Since  $K_\tau \subset G$ , then  $\text{cap}(K_\tau; B_{p,q}^\alpha) \leq \text{cap}(G; B_{p,q}^\alpha)$ , and we have

$$(1 - \alpha_\nu)^{1/q} \|f_\nu\|_{b_{p,q}^{\alpha_\nu}} \leq \Lambda(\alpha_\nu) + \frac{1}{\nu}. \quad (3.13)$$

We shall estimate  $\omega_j(f_\nu; \delta)_p$ . Using (3.13) and Lemma 2.6 with  $\theta = \infty$ , we obtain that

$$\Lambda(\alpha_\nu) + \frac{1}{\nu} \geq q^{-1/q} \frac{1 + \alpha_\nu}{2} \delta^{-\alpha_\nu q} \sum_{j=1}^n \omega_j(f_\nu; \delta)_p \quad (3.14)$$

for any  $\delta > 0$  and any  $\nu \in \mathbb{N}$ . In particular, (3.14) and (3.12) yield that

$$\sum_{k=1}^n \omega_k(f_\nu; \delta)_p \leq A \delta^{\alpha_\nu}, \quad \delta > 0, \quad (3.15)$$

where  $A = 2q^{1/q}(\lim_{\alpha \rightarrow 1-0} \Lambda(\alpha) + 2)$  depends only on  $p, q, n$ , and  $G$ . To get also a control of  $L^p$ -norms, we apply truncation to the functions  $f_\nu$ . Let  $0 < \varepsilon < 1/2$ . Set

$$E_{\nu,\varepsilon} = \{x \in \mathbb{R}^n : f_\nu(x) > \varepsilon\}.$$

Let  $p^* = np/(n-p)$  and  $p_\nu = np/(n - \alpha_\nu p)$ ; then  $p_\nu < p^*$ . By Lemma 2.8,

$$|E_{\nu,\varepsilon}| \leq (2p_\nu)^{p_\nu} \varepsilon^{-p_\nu} \|f_\nu\|_{b_{p,\infty}^{\alpha_\nu}}^{p_\nu} \leq (2p^*)^{p^*} \varepsilon^{-p^*} \|f_\nu\|_{b_{p,\infty}^{\alpha_\nu}}^{p_\nu}.$$

Thus, using (3.15), we obtain

$$|E_{\nu,\varepsilon}| \leq A' \varepsilon^{-p^*} \quad (\nu \in \mathbb{N}), \quad (3.16)$$

where  $A'$  depends only on  $p, q, n$ , and  $G$ ,  $A' = (2p^*A)^{p^*}$ . Set now

$$f_{\nu,\varepsilon}(x) = \frac{1}{1-\varepsilon} \max(f_\nu(x) - \varepsilon, 0).$$

It is easily seen that

$$\omega_j(f_{\nu,\varepsilon}; \delta)_p \leq \frac{1}{1-\varepsilon} \omega_j(f_\nu; \delta)_p, \quad \delta \geq 0 \quad (j = 1, \dots, n). \quad (3.17)$$

Moreover,  $0 \leq f_{\nu,\varepsilon}(x) \leq 1$  for all  $x \in \mathbb{R}^n$ ,  $f_{\nu,\varepsilon}(x) = 1$  for all  $x \in K_\tau$ , and  $f_{\nu,\varepsilon}(x) = 0$  for  $x \notin E_{\nu,\varepsilon}$ . Applying (3.16), we get

$$\|f_{\nu,\varepsilon}\|_p^p \leq |E_{\nu,\varepsilon}| \leq A' \varepsilon^{-p^*} \quad (\nu \in \mathbb{N}). \quad (3.18)$$

Besides, by (3.15) and (3.17),

$$\omega(f_{\nu,\varepsilon}; \delta)_p \leq 2A' \delta^{\alpha_1}, \quad \delta \in [0, 1], \quad \nu \in \mathbb{N}. \quad (3.19)$$

By virtue of (3.18), (3.19), and the compactness criterion (see [8, p. 111]), for any compact set  $Q \subset \mathbb{R}^n$  there exists a subsequence of  $\{f_{\nu,\varepsilon}\}$  that converges in  $L^p(Q)$ . Therefore, by Riesz's theorem, for any compact set  $Q \subset \mathbb{R}^n$  there exists a subsequence of  $\{f_{\nu,\varepsilon}\}$  that converges almost everywhere on  $Q$ . Let  $Q_s = [-s, s]^n$ ,  $s \in \mathbb{N}$ . A successive extraction of subsequences gives strictly increasing sequences  $\{\nu_m^{(s)}\}$  ( $s = 1, 2, \dots$ ) of natural numbers such that

$$\{\nu_m^{(1)}\} \supset \{\nu_m^{(2)}\} \supset \dots \supset \{\nu_m^{(s)}\} \supset \dots$$

and for each  $s \in \mathbb{N}$  the subsequence  $\{f_{\nu_m^{(s)},\varepsilon}\}$  converges almost everywhere on  $Q_s$ . Then the diagonal subsequence  $\{f_{\nu_m^{(s)},\varepsilon}\}$  converges almost everywhere on  $\mathbb{R}^n$ . For simplicity, we assume that  $\{f_{\nu,\varepsilon}\}$  itself converges almost everywhere on  $\mathbb{R}^n$ . Let

$$f_\varepsilon(x) = \lim_{\nu \rightarrow \infty} f_{\nu,\varepsilon}(x).$$

Since  $f_{\nu,\varepsilon}(x) = 1$  on  $K_\tau$  for any  $\nu \in \mathbb{N}$ , then

$$f_\varepsilon(x) = 1 \quad \text{for all } x \in K_\tau. \quad (3.20)$$

We have also that  $0 \leq f_\varepsilon(x) \leq 1$  almost everywhere on  $\mathbb{R}^n$ . Further, by Fatou's lemma and (3.18)

$$\|f_\varepsilon\|_p^p \leq A' \varepsilon^{-p^*}. \quad (3.21)$$

Fatou's lemma yields also that for any  $h > 0$  and any  $j = 1, \dots, n$

$$\int_{\mathbb{R}^n} |f_\varepsilon(x + he_j) - f_\varepsilon(x)|^p dx \leq \liminf_{\nu \rightarrow \infty} \int_{\mathbb{R}^n} |f_{\nu,\varepsilon}(x + he_j) - f_{\nu,\varepsilon}(x)|^p dx.$$



Thus,

$$\omega_j(f_\varepsilon; \delta)_p \leq \lim_{\nu \rightarrow \infty} \omega_j(f_{\nu, \varepsilon}; \delta)_p, \quad \delta \geq 0 \quad (j = 1, \dots, n). \quad (3.22)$$

Let  $\varphi_\tau$  be the mollifier defined by (2.21). Set  $f_{\varepsilon, \tau} = f_\varepsilon * \varphi_\tau$ . Clearly,  $0 \leq f_{\varepsilon, \tau}(x) \leq 1$  for all  $x \in \mathbb{R}^n$  and, by (2.22) and (3.20),

$$f_{\varepsilon, \tau}(x) = 1 \quad \text{if} \quad \text{dist}(x, K) < \tau. \quad (3.23)$$

Besides, by Young inequality and (2.22),

$$\omega_j(f_{\varepsilon, \tau}; \delta)_p \leq \omega_j(f_\varepsilon; \delta)_p, \quad \delta \geq 0 \quad (j = 1, \dots, n). \quad (3.24)$$

Applying inequalities (3.14) and (3.17), we obtain

$$\Lambda(\alpha_\nu) + \frac{1}{\nu} \geq \frac{(1 + \alpha_\nu)(1 - \varepsilon)}{2q^{1/q}} \delta^{-\alpha_\nu} \sum_{j=1}^n \omega_j(f_{\nu, \varepsilon}; \delta)_p.$$

By (3.11), (3.22), and (3.24), this implies that

$$\lim_{\alpha \rightarrow 1-0} \Lambda(\alpha) \geq \frac{1 - \varepsilon}{q^{1/q}} \sum_{j=1}^n \frac{\omega_j(f_{\varepsilon, \tau}; \delta)_p}{\delta} \quad (3.25)$$

for any  $\delta > 0$ . Taking into account (3.21), we have  $f_{\varepsilon, \tau} \in L^p(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ . Making  $\delta$  tend to zero and applying Lemma 2.2, we obtain

$$\lim_{\alpha \rightarrow 1-0} \Lambda(\alpha) \geq \frac{1 - \varepsilon}{q^{1/q}} \sum_{j=1}^n \|D_j f_{\varepsilon, \tau}\|_p. \quad (3.26)$$

Let  $\eta$  be the cutoff function defined by (2.23)). Set  $g(x) = f_{\varepsilon, \tau}(x)\eta(\gamma x)$ ,  $\gamma > 0$ . Then  $g \in C_0^\infty(\mathbb{R}^n)$  and  $0 \leq g(x) \leq 1$  for all  $x \in \mathbb{R}^n$ . If  $\gamma$  is sufficiently small, then, by virtue of (3.23),  $g(x) = 1$  if  $\text{dist}(x, K) < \tau$ . Moreover,  $\gamma$  can be chosen so small that (see (2.24))

$$\|D_j g\|_p < \|D_j f_{\varepsilon, \tau}\|_p + \frac{\varepsilon}{n} \quad (j = 1, \dots, n).$$

Since

$$\sum_{j=1}^n \|D_j g\|_p \geq \text{cap}(K; W_p^1)^{1/p},$$

inequality (3.26) yields that

$$\lim_{\alpha \rightarrow 1-0} \Lambda(\alpha) \geq \frac{1 - \varepsilon}{q^{1/q}} [\text{cap}(K; W_p^1)^{1/p} - \varepsilon].$$

Taking into account that  $\varepsilon \in (0, 1)$  and a compact set  $K \subset G$  are arbitrary, we obtain inequality (3.9). Together with (3.3), this gives (3.1).  $\square$

**Remark 3.2.** The statement of Theorem 3.1 fails to hold for *compact* sets. To show it, we use a theorem on capacity of a Cantor set [2, Section 5.3]. Let  $1 < p < n$ ,  $p = q$ , and let  $0 < \alpha < 1$ . It is known that in this case the  $B_p^\alpha$ -capacity is equivalent to the Bessel capacity  $C_{\alpha,p}$  [2, p. 107]. Set

$$l_k = ((k+4)^2 2^{-kn})^{1/(n-p)} \quad (k = 0, 1, \dots).$$

Then  $l_{k+1} < l_k/2$  for all  $k \geq 0$ . Further,

$$\sum_{k=0}^{\infty} 2^{-kn} l_k^{p-n} < \infty \quad \text{and} \quad \sum_{k=0}^{\infty} 2^{-kn} l_k^{\alpha p-n} = \infty$$

for any  $0 < \alpha < 1$ . Let  $E$  be the Cantor set corresponding to the sequence  $\{l_k\}$  defined in [2, (5.3.1)]. It follows by [2, Theorem 5.3.2] that

$$\text{cap}(E; W_p^1) > 0 \quad \text{and} \quad \text{cap}(E; B_p^\alpha) = 0$$

for any  $0 < \alpha < 1$ . Thus, equality (3.1) does not hold.

**Remark 3.3.** We observe that if  $n < p < \infty$ ,  $n \in \mathbb{N}$ , or  $p = n \geq 2$ , then equality (3.1) is trivially true. It is closely related to the fact that in these cases the Sobolev capacity of a ball in  $\mathbb{R}^n$  is equal to zero (see [20, p. 148]). For completeness, we give the corresponding arguments in detail.

First, let  $n < p < \infty$ . We consider the ball  $\mathcal{B}_r$ ,  $r > 0$ . Let  $\eta$  be the cutoff function defined by (2.23). Set  $f_\gamma(x) = \eta(\gamma x)$ , where  $0 < \gamma < 1/r$ . Then  $f_\gamma \in C_0^\infty(\mathbb{R}^n)$ ,  $0 \leq f_\gamma(x) \leq 1$  for all  $x \in \mathbb{R}^n$ , and  $f_\gamma(x) = 1$  in some neighborhood of  $\overline{\mathcal{B}_r}$ . Further,

$$\|D_k f_\gamma\|_p = \gamma^{1-n/p} \|D_k \eta\|_p \quad (k = 1, \dots, n). \quad (3.27)$$

This implies that  $\text{cap}(\mathcal{B}_r; W_p^1) = 0$ . Moreover, if  $n/p < \alpha < 1$ , then we have also that

$$\text{cap}(\mathcal{B}_r; B_{p,q}^\alpha) = 0 \quad (3.28)$$

for any  $1 \leq q < \infty$ . Indeed,

$$\text{cap}(\mathcal{B}_r; B_{p,q}^\alpha) \leq \|f_\gamma\|_{b_{p,q}^\alpha}^p.$$

Thus, applying Lemma 2.4 and (3.27), we obtain

$$\begin{aligned} \|f_\gamma\|_{b_{p,q;k}^\alpha} &\leq q^{-1/q} [(1-\alpha)^{-1/q} T^{1-\alpha} \|D_k f_\gamma\|_p + 2\alpha^{-1/q} T^{-\alpha} \|f_\gamma\|_p] \\ &\leq ((1-\alpha)q)^{-1/q} \|D_k \eta\|_p T^{1-\alpha} \gamma^{1-n/p} + 2(\alpha q)^{-1/q} \|\eta\|_p T^{-\alpha} \gamma^{-n/p} \end{aligned}$$

for any  $T > 0$  and any  $1 \leq k \leq n$ . Setting  $T = 1/\gamma$ , we get

$$\|f_\gamma\|_{b_{p,q;k}^\alpha} \leq [((1-\alpha)q)^{-1/q} \|D_k \eta\|_p + 2(\alpha q)^{-1/q} \|\eta\|_p] \gamma^{\alpha-n/p}.$$

Since  $0 < \gamma < 1/r$  is arbitrary and  $\alpha > n/p$ , this implies (3.28). Thus, if  $p > n$ , then for any open set  $G \subset \mathbb{R}^n$  both the capacities in relation (3.1) are equal to 0.

Let now  $p = n \geq 2$ . We have  $\text{cap}(\mathcal{B}_r; W_n^1) = 0$  ( $r > 0$ ) [20, p. 148]. At the same time, it follows from Lemma 2.8 and inequality (2.9) that  $\text{cap}(\mathcal{B}_r; B_{n,q}^\alpha) > 0$  for any  $0 < \alpha < 1$  and any  $1 \leq q < \infty$ . Nevertheless, we shall show that

$$\lim_{\alpha \rightarrow 0} (1 - \alpha)^{n/q} \text{cap}(\mathcal{B}_r; B_{n,q}^\alpha) = 0 \quad (r > 0). \quad (3.29)$$

Let  $\sigma = (n - 1)/(2n)$  and set

$$f_0(x) = \begin{cases} |\ln |x||^\sigma & \text{if } |x| \leq 1 \\ 0 & \text{if } |x| > 1. \end{cases}$$

It is easy to see that  $f \in W_n^1(\mathbb{R}^n)$ . Let  $\varepsilon > 0$ . Set  $f_1(x) = \min(\varepsilon f_0(x), 1)$ . Since  $f_0(x) \rightarrow +\infty$  as  $x \rightarrow 0$ , there exists a closed ball  $U_\varepsilon$  centered at the origin such that  $f_1(x) = 1$  for all  $x \in U_\varepsilon$ . There is  $\gamma > 0$  such that  $\gamma x \in U_\varepsilon$  for all  $x \in \overline{\mathcal{B}_{r+1}}$ . Set  $f_2(x) = f_1(\gamma x)$ . Then

$$\|D_k f_2\|_n = \|D_k f_1\|_n \leq \varepsilon \|D_k f_0\|_n \quad (k = 1, \dots, n)$$

and

$$\|f_2\|_n = \frac{\|f_1\|_n}{\gamma} \leq \frac{\varepsilon \|f_0\|_n}{\gamma}.$$

Finally, we define  $f = f_2 * \varphi_{1/2}$  (see (2.21)). Then  $f \in C_0^\infty(\mathbb{R}^n)$ ,  $f(x) = 1$  in  $\mathcal{B}_{r+1/2}$ , and  $0 \leq f(x) \leq 1$  for all  $x \in \mathbb{R}^n$ . Moreover,

$$\|D_k f\|_n \leq \varepsilon \|D_k f_0\|_n \quad (k = 1, \dots, n) \quad (3.30)$$

and

$$\|f\|_n \leq \frac{\varepsilon \|f_0\|_n}{\gamma}. \quad (3.31)$$

First, this shows that  $\text{cap}(\mathcal{B}_r; W_n^1) = 0$ . Further, applying Lemma 2.4 with  $T = 1$  and using (3.30) and (3.31), we obtain

$$\begin{aligned} (1 - \alpha)^{1/q} \|f\|_{b_{n,q;k}^\alpha} &\leq q^{-1/q} \left( \|D_k f\|_n + 2 \left( \frac{1 - \alpha}{\alpha} \right)^{1/q} \|f\|_n \right) \\ &\leq \varepsilon q^{-1/q} \left( \|D_k f_0\|_n + \frac{2}{\gamma} \left( \frac{1 - \alpha}{\alpha} \right)^{1/q} \|f_0\|_n \right). \end{aligned}$$

Since  $\text{cap}(\mathcal{B}_r; B_{n,q}^\alpha) \leq \|f_\gamma\|_{b_{n,q}^\alpha}^n$ , this implies that

$$\overline{\lim}_{\alpha \rightarrow 0} (1 - \alpha)^{1/q} \text{cap}(\mathcal{B}_r; B_{n,q}^\alpha)^{1/n} \leq \varepsilon q^{-1/q} \sum_{k=1}^n \|D_k f_0\|_n.$$

By view of the arbitrariness of  $\varepsilon > 0$ , we obtain (3.29). Thus, for  $p = n \geq 2$  (3.1) also is trivially true.

**Remark 3.4.** The remaining case  $p = n = 1$  is also "degenerate". First, if a set  $E$  consists of one point,  $E = \{x_0\}$ , then  $\text{cap}(E; W_1^1) \geq 2$ . Indeed, if  $f \in C_0^\infty(\mathbb{R})$  and  $f(x_0) = 1$ , then

$$-\int_{x_0}^{\infty} f'(x) dx = \int_{-\infty}^{x_0} f'(x) dx = 1.$$

Thus,  $\|f'\|_1 \geq 2$ . Further, let  $K \subset \mathbb{R}$  be an arbitrary compact set,  $K \subset [-a, a]$  ( $a > 0$ ). Set

$$f_a(x) = \begin{cases} 1 & \text{if } |x| \leq a \\ (a/x)^2 & \text{if } |x| > a. \end{cases} \quad (3.32)$$

Then  $f_a \in W_1^1(\mathbb{R})$  and  $\|f'_a\|_1 = 2$ . We obtain that  $\text{cap}(K; W_1^1) = 2$  for any compact set  $K \neq \emptyset$ , and therefore  $\text{cap}(G; W_1^1) = 2$  for any non-empty open set  $G \subset \mathbb{R}$ .

Now we observe that for any  $f \in L^1(\mathbb{R})$  and any  $h > 0$

$$\begin{aligned} & \int_0^{\infty} |f(x) - f(x+h)| dx \\ & \geq \int_0^{\infty} |f(x)| dx - \int_0^{\infty} |f(x+h)| dx = \int_0^h |f(x)| dx, \end{aligned}$$

and similarly

$$\int_{-\infty}^0 |f(x) - f(x+h)| dx \geq \int_0^h |f(x)| dx.$$

Thus,

$$\omega(f; h)_1 \geq 2 \int_0^h |f(x)| dx \quad (h > 0). \quad (3.33)$$

Let  $I = [-h_0, h_0]$  ( $h_0 > 0$ ). Let  $f \in L^1(\mathbb{R})$  and  $f(x) = 1$  on  $I$ . Then, by (3.33),  $\omega(f; h)_1 \geq 2h$  for all  $0 \leq h \leq h_0$ . Thus, for any  $1 \leq q < \infty$

$$\begin{aligned} (1 - \alpha) \|f\|_{b_{1,q}^\alpha}^q & \geq (1 - \alpha) \int_0^{h_0} h^{-\alpha q} \omega(f; h)_1^q \frac{dh}{h} \\ & \geq 2^q (1 - \alpha) \int_0^{h_0} h^{(1-\alpha)q} \frac{dh}{h} = \frac{2^q}{q} h_0^{(1-\alpha)q}. \end{aligned}$$

This implies that

$$\lim_{\alpha \rightarrow 1-} (1 - \alpha)^{1/q} \text{cap}(G; B_{1,q}^\alpha) \geq 2q^{-1/q}$$

for any open set  $G \subset \mathbb{R}$ . On the other hand, assume that  $G \subset [-a, a]$  ( $a > 0$ ). Applying Lemma 2.4 to the function (3.32), we have

$$\begin{aligned} (1 - \alpha)^{1/q} \|f_a\|_{b_{1,q}^\alpha} &\leq q^{-1/q} \|f'_a\|_1 + 2 \left( \frac{1 - \alpha}{\alpha q} \right)^{1/q} \|f_a\|_1 \\ &= q^{-1/q} \left[ 2 + 8a \left( \frac{1 - \alpha}{\alpha} \right)^{1/q} \right]. \end{aligned}$$

It follows that

$$\overline{\lim}_{\alpha \rightarrow 1^-} (1 - \alpha)^{1/q} \text{cap}(G; B_{1,q}^\alpha) \leq 2q^{-1/q}.$$

Thus, for any open bounded set  $G \subset \mathbb{R}$

$$\lim_{\alpha \rightarrow 1^-} (1 - \alpha)^{1/q} \text{cap}(G; B_{1,q}^\alpha) = q^{-1/q} \text{cap}(G; W_1^1) = 2q^{-1/q}.$$

#### 4. THE LIMIT AS $\alpha \rightarrow 0$

In this section we study the behaviour of  $B_{p,q}^\alpha$ -capacities as  $\alpha \rightarrow 0$  (cf. (1.6) and Remark 4.3 below).

**Theorem 4.1.** *Let  $1 \leq p < \infty$  and  $1 \leq q < \infty$ . Then for any compact set  $K \subset \mathbb{R}^n$*

$$\lim_{\alpha \rightarrow 0^+} \alpha^{p/q} \text{cap}(K; B_{p,q}^\alpha) = 2n^p \left( \frac{1}{q} \right)^{p/q} |K|. \quad (4.1)$$

*Proof.* Denote

$$\Lambda(\alpha) = \alpha^{1/q} [\text{cap}(K; B_{p,q}^\alpha)]^{1/p}, \quad 0 < \alpha < 1.$$

First we prove that

$$\underline{\lim}_{\alpha \rightarrow 0^+} \Lambda(\alpha) \geq n 2^{1/p} q^{-1/q} |K|^{1/p}. \quad (4.2)$$

We assume that  $|K| > 0$ . It is clear that  $\underline{\lim}_{\alpha \rightarrow 0^+} \Lambda(\alpha) < \infty$ . There exists a decreasing sequence  $\{\alpha_\nu\}$  of numbers  $\alpha_\nu \in (0, 1/2]$  with  $\alpha_1 = \min(1, n/p)/2$  such that  $\alpha_\nu \rightarrow 0$  and

$$\lim_{\nu \rightarrow \infty} \Lambda(\alpha_\nu) = \underline{\lim}_{\alpha \rightarrow 0^+} \Lambda(\alpha). \quad (4.3)$$

We emphasize that  $\alpha_\nu < n/p$  for all  $\nu \in \mathbb{N}$ . We may assume that

$$\Lambda(\alpha_\nu) \leq \underline{\lim}_{\alpha \rightarrow 0^+} \Lambda(\alpha) + 1 \quad (\nu \in \mathbb{N}). \quad (4.4)$$

For any  $\nu \in \mathbb{N}$  there exists a function  $f_\nu \in C_0^\infty(\mathbb{R}^n)$  such that  $0 \leq f_\nu(x) \leq 1$  for all  $x \in \mathbb{R}^n$ ,  $f_\nu(x) = 1$  for all  $x \in K$ , and

$$\Lambda(\alpha_\nu) \geq \alpha_\nu^{1/q} \|f_\nu\|_{b_{p,q}^{\alpha_\nu}} - \frac{1}{\nu}.$$

Applying Lemma 2.5 for  $\theta = \infty$ , we obtain that

$$\Lambda(\alpha_\nu) + \frac{1}{\nu} \geq q^{-1/q} t^{-\alpha_\nu} \sum_{j=1}^n \omega_j(f_\nu; t)_p \quad (4.5)$$

for any  $t > 0$  and any  $\nu \in \mathbb{N}$ . In particular, by (4.4) and (4.5),

$$\sum_{j=1}^n \omega_j(f_\nu; t)_p \leq A t^{\alpha_\nu}, \quad t > 0, \quad (4.6)$$

where  $A$  depends only on  $p, q, n$ , and  $K$ .

Let  $0 < \varepsilon < 1$ . Set

$$E_{\nu, \varepsilon} = \{x \in \mathbb{R}^n : f_\nu(x) > \varepsilon\}.$$

Denote  $p_\nu = np/(n - \alpha_\nu p)$ . Then  $p_\nu \leq p_1$ . By Lemma 2.8,

$$|E_{\nu, \varepsilon}| \leq (2p_\nu)^{p_\nu} \varepsilon^{-p_\nu} \|f_\nu\|_{b_{p, \infty}^{\alpha_\nu}}^{p_\nu} \leq (2p_1)^{p_1} \varepsilon^{-p_1} \|f_\nu\|_{b_{p, \infty}^{\alpha_\nu}}^{p_\nu}.$$

Thus, using (4.6), we obtain that

$$|E_{\nu, \varepsilon}| \leq A' \varepsilon^{-p_1} \quad (\nu \in \mathbb{N}), \quad (4.7)$$

where  $A'$  depends only on  $p, q, n$ , and  $K$ .

Since  $|K| > 0$ , there exists a number  $\lambda(K) > 0$  such that

$$\text{mes}_{n-1} \Pi_j(K) \geq \lambda(K) \quad \text{for all } 1 \leq j \leq n.$$

Further,  $K \subset E_{\nu, \varepsilon}$ , and thus

$$\text{mes}_{n-1} \Pi_j(E_{\nu, \varepsilon}) \geq \lambda(K) \quad (\nu \in \mathbb{N}, j = 1, \dots, n).$$

Now we apply Lemma 2.9 with  $\mu = A' \varepsilon^{-p_1}$ ,  $\lambda = \lambda(K)$ , and  $\eta = \varepsilon|K|$ . Set  $H = 2\mu^2 n / (\lambda \eta)$ . By Lemma 2.9, for any  $\nu \in \mathbb{N}$  there exists  $h_\nu \in (0, H]$  such that

$$\sum_{j=1}^n |\{x \in E_{\nu, \varepsilon} : x + h_\nu e_j \in E_{\nu, \varepsilon}\}| < \varepsilon|K|. \quad (4.8)$$

We emphasize that  $H$  doesn't depend on  $\nu$ . Denote

$$K_j^{(\nu)} = \{x \in \mathbb{R}^n : x + h_\nu e_j \in K\}.$$

Since  $K \subset E_{\nu, \varepsilon}$  ( $\nu \in \mathbb{N}$ ), we derive from (4.8) that for any  $\nu \in \mathbb{N}$  and any  $j = 1, \dots, n$

$$|\{x \in K : f_\nu(x + h_\nu e_j) \leq \varepsilon\}| > (1 - \varepsilon)|K|,$$

$$|\{x \in K_j^{(\nu)} : f_\nu(x) \leq \varepsilon\}| > (1 - \varepsilon)|K|,$$

and  $|K \cap K_j^{(\nu)}| < \varepsilon|K|$ . Thus, taking into account that  $0 \leq f_\nu(x) \leq 1$  and  $f_\nu(x) = 1$  on  $K$ , we obtain

$$\begin{aligned} \omega_j(f_\nu; H)_p^p &\geq \int_K |f_\nu(x) - f_\nu(x + h_\nu e_j)|^p dx \\ &+ \int_{K_j^{(\nu)}} |f_\nu(x) - f_\nu(x + h_\nu e_j)|^p dx - \varepsilon|K| \geq 2(1 - \varepsilon)^{p+1}|K| - \varepsilon|K| \end{aligned}$$

for all  $1 \leq j \leq n$ . From here and (4.5),

$$\begin{aligned} \Lambda(\alpha_\nu) + \frac{1}{\nu} &\geq q^{-1/q} H^{-\alpha_\nu} \sum_{j=1}^n \omega_j(f_\nu; H)_p \\ &\geq nq^{-1/q} H^{-\alpha_\nu} [(2(1 - \varepsilon)^{p+1} - \varepsilon)|K|]^{1/p}. \end{aligned}$$

By (4.3), this implies that

$$\lim_{\alpha \rightarrow 0+} \Lambda(\alpha) \geq nq^{-1/q} [(2(1 - \varepsilon)^{p+1} - \varepsilon)|K|]^{1/p}.$$

Since  $\varepsilon \in (0, 1)$  is arbitrary, this yields (4.2).

Now we shall prove that

$$\overline{\lim}_{\alpha \rightarrow 0+} \Lambda(\alpha) \leq n2^{1/p} q^{-1/q} |K|^{1/p}. \quad (4.9)$$

Set for  $\tau > 0$

$$K_\tau = \{x \in \mathbb{R}^n : \text{dist}(x, K) \leq 2\tau\}. \quad (4.10)$$

Fix  $0 < \varepsilon < 1$  and choose  $\tau > 0$  such that

$$|K_\tau| < |K| + \varepsilon. \quad (4.11)$$

Let  $\varphi_\tau$  be the standard mollifier defined by (2.20). Set

$$f_\tau = \chi_\tau * \varphi_\tau,$$

where  $\chi_\tau$  is the characteristic function of the set  $K_\tau$ . Then  $f_\tau \in C_0^\infty(\mathbb{R}^n)$ ,  $0 \leq f_\tau(x) \leq 1$  for all  $x \in \mathbb{R}^n$ , and  $f_\tau(x) = 1$  for all  $x$  such that  $\text{dist}(x, K) \leq \tau$ . Thus,

$$\text{cap}(K; B_{p,q}^\alpha) \leq \|f_\tau\|_{b_{p,q}^\alpha}^p. \quad (4.12)$$

Using (2.6) and (2.22), we have

$$\omega_j(f_\tau; t)_p \leq \omega_j(\chi_\tau; t)_p \leq (2|K_\tau|)^{1/p} \leq [2(|K| + \varepsilon)]^{1/p}. \quad (4.13)$$

Applying (2.7) and (4.13), we obtain

$$\begin{aligned}
\alpha^{1/q} \|f_\tau\|_{b_{p,q}^\alpha} &= \alpha^{1/q} \sum_{j=1}^n \left( \int_0^\infty t^{-\alpha q} \omega_j(f_\tau; t)_p^q \frac{dt}{t} \right)^{1/q} \\
&\leq \alpha^{1/q} \sum_{j=1}^n \left[ \|D_j f_\tau\|_p \left( \int_0^1 t^{(1-\alpha)q} \frac{dt}{t} \right)^{1/q} + \left( \int_1^\infty t^{-\alpha q} \omega_j(f_\tau; t)_p^q \frac{dt}{t} \right)^{1/q} \right] \\
&\leq \left( \frac{\alpha}{(1-\alpha)q} \right)^{1/q} \sum_{j=1}^n \|D_j f_\tau\|_p + n 2^{1/p} q^{-1/q} (|K| + \varepsilon)^{1/p}.
\end{aligned}$$

This estimate and (4.12) imply that

$$\Lambda(\alpha) \leq \left( \frac{\alpha}{(1-\alpha)q} \right)^{1/q} \sum_{j=1}^n \|D_j f_\tau\|_p + 2^{1/p} q^{-1/q} n (|K| + \varepsilon)^{1/p}.$$

It follows that

$$\overline{\lim}_{\alpha \rightarrow 0+} \Lambda(\alpha) \leq n 2^{1/p} q^{-1/q} (|K| + \varepsilon)^{1/p}.$$

Since  $\varepsilon \in (0, 1)$  is arbitrary, this implies (4.9). Inequalities (4.2) and (4.9) yield (4.1).  $\square$

**Remark 4.2.** Generally, Theorem 4.1 fails to hold for *open* sets. As in Section 3, we shall show it using Cantor sets [2, Section 5.3].

Let  $f \in B_p^\alpha(\mathbb{R}^n)$  and let  $\delta_\lambda f(x) = f(\lambda x)$  ( $\lambda > 0$ ) be a dilation of  $f$ . It is easily seen that

$$\|\delta_\lambda f\|_{b_p^\alpha}^p = \lambda^{\alpha p - n} \|f\|_{b_p^\alpha}^p. \quad (4.14)$$

Assume that  $p > 1$  and  $0 < \alpha < \min(1, n/p)$ . Recall that in this case the  $B_p^\alpha$ -capacity is equivalent to the Bessel capacity  $C_{\alpha,p}$  [2, p. 107]. There exists  $k_0 = k_0(\alpha)$  such that the sequence

$$l_k = (2^{-kn}(k + k_0)^2)^{1/(n-\alpha p)}$$

satisfies the condition  $l_{k+1} \leq l_k/2$  ( $k = 0, 1, \dots$ ). Moreover,

$$\sum_{k=0}^{\infty} 2^{-kn} l_k^{\alpha p - n} < \infty.$$

Let  $K_\alpha$  be the Cantor set corresponding to the sequence  $\{l_k\}$ , defined in [2, (5.3.1)]. Then  $|K_\alpha| = 0$  and by [2, Theorem 5.3.2],  $\text{cap}(K_\alpha; B_p^\alpha) > 0$ . For  $\lambda > 0$ , set

$$K_{\alpha,\lambda} = \{x \in \mathbb{R}^n : \frac{x}{\lambda} \in K_\alpha\}.$$



There exists a function  $f_{\alpha,\lambda} \in C_0^\infty$  such that  $0 \leq f_{\alpha,\lambda}(x) \leq 1$  for all  $x \in \mathbb{R}^n$ ,  $f_{\alpha,\lambda}(x) = 1$  in some neighborhood of  $K_{\alpha,\lambda}$ , and

$$\|f_{\alpha,\lambda}\|_{b_p^\alpha}^p \leq \text{cap}(K_{\alpha,\lambda}; B_p^\alpha) + 1.$$

Set  $g_{\alpha,\lambda}(x) = f_{\alpha,\lambda}(\lambda x)$ . Then  $g_{\alpha,\lambda}(x) = 1$  in some neighborhood of  $K_\alpha$ . Thus, using (4.14), we obtain

$$\begin{aligned} \text{cap}(K_\alpha; B_p^\alpha) &\leq \|g_{\alpha,\lambda}\|_{b_p^\alpha}^p = \lambda^{\alpha p - n} \|f_{\alpha,\lambda}\|_{b_p^\alpha}^p \\ &\leq \lambda^{\alpha p - n} (\text{cap}(K_{\alpha,\lambda}; B_p^\alpha) + 1). \end{aligned}$$

From here,

$$\text{cap}(K_{\alpha,\lambda}; B_p^\alpha) \geq \lambda^{n - \alpha p} \text{cap}(K_\alpha; B_p^\alpha) - 1.$$

Since  $\text{cap}(K_\alpha; B_p^\alpha) > 0$ , we can choose such  $\lambda(\alpha) > 0$  that

$$\alpha \text{cap}(K_{\alpha,\lambda(\alpha)}; B_p^\alpha) > 1.$$

Thus, for any  $0 < \alpha < \min(1, n/p)$  there exists a compact set  $E_\alpha$  such that

$$|E_\alpha| = 0 \quad \text{and} \quad \alpha \text{cap}(E_\alpha; B_p^\alpha) > 1.$$

Let  $j_0 = [(\min(1, n/p))^{-1}] + 1$ . Set  $E_j^* = E_{1/j}$ ,  $j \geq j_0$ . Then

$$\alpha \text{cap}(E_j^*; B_p^\alpha) > 1 \quad \text{for} \quad \alpha = \frac{1}{j} \quad (j \geq j_0).$$

Further, set  $E = \cup_{j=j_0}^\infty E_j^*$ . Then  $|E| = 0$ . Let  $0 < \varepsilon < 1$ . There exists an open set  $G$  such that  $E \subset G$  and  $|G| < \varepsilon$ . We have

$$\alpha \text{cap}(G; B_p^\alpha) \geq \alpha \text{cap}(E_j^*; B_p^\alpha) > 1 \quad \text{for} \quad \alpha = \frac{1}{j} \quad (j \geq j_0).$$

Thus,

$$\overline{\lim}_{\alpha \rightarrow 0} \alpha \text{cap}(G; B_p^\alpha) \geq 1,$$

and equality (4.1) does not hold for the set  $G$ .

**Remark 4.3.** Our final remark concerns limiting relation (1.6). This relation was proved in [21] for the seminorm

$$\left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x+h) - f(x)|^p}{|h|^{n+\alpha p}} dx dh \right)^{1/p}.$$

It is well known that this seminorm is equivalent to  $\|f\|_{b_p^\alpha}$ . We shall briefly discuss the limiting behaviour of  $\alpha \|f\|_{b_p^\alpha}$ .

Assume that a function  $f$  belongs to  $B_{p,q}^{\alpha_0}(\mathbb{R}^n)$  for some  $0 < \alpha_0 < 1$ . Then  $f \in B_{p,q}^\alpha(\mathbb{R}^n)$  for any  $0 < \alpha \leq \alpha_0$ . Moreover, it follows immediately from [10, Lemma 1] that

$$\lim_{\alpha \rightarrow 0} \alpha^{1/q} \|f\|_{b_{p,q}^\alpha} = q^{-1/q} \sum_{j=1}^n \omega_j(f; +\infty)_p. \quad (4.15)$$

It is also easily seen that

$$\lim_{h \rightarrow \infty} \int_{\mathbb{R}^n} |f(x + he_j) - f(x)|^p dx = 2 \|f\|_p^p \quad (j = 1, \dots, n).$$

This equality and (2.5) imply that for a *nonnegative*  $f$

$$\omega_j(f; +\infty) = 2^{1/p} \|f\|_p \quad (j = 1, \dots, n) \quad (4.16)$$

and thus by (4.15)

$$\lim_{\alpha \rightarrow 0} \alpha^{1/q} \|f\|_{b_{p,q}^\alpha} = q^{-1/q} 2^{1/p} n \|f\|_p \quad \text{if } f \geq 0. \quad (4.17)$$

However, equalities (4.16) and (4.17) fail to hold in a general case. We consider the following simple example for  $n = 1$ . Let  $I_k = [k, k+1)$  ( $k = 0, 1, \dots, 2\nu$ ). Set

$$f_\nu(x) = \sum_{k=0}^{2\nu} (-1)^k \chi_{I_k}(x).$$

Then  $\|f_\nu\|_p = (2\nu + 1)^{1/p}$ . Further,

$$\begin{aligned} & \int_{\mathbb{R}} |f_\nu(x+1) - f_\nu(x)|^p dx \\ & \geq \sum_{k=0}^{2\nu-1} \int_{I_k} |f_\nu(x+1) - f_\nu(x)|^p dx = 2^{p+1} \nu. \end{aligned}$$

Thus,

$$\omega(f_\nu; 1)_p \geq 2 \left( \frac{2\nu}{2\nu+1} \right)^{1/p} \|f_\nu\|_p.$$

It shows that the constant 2 on the right-hand side of (2.4) is optimal, and thus (4.16) and (4.17) may not be true.

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